Iterative Method for Calculating Derivatives of Eigenvectors

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Introduction

THE partial derivatives of eigenvalues and eigenvectors of structures with respect to design variables or system parameters have a wide application area, such as optimal dynamic design, parameter identification, model modification, and system control. The most straightforward approach for calculating the derivatives is the finite difference method. Fox and Kapoor[1] derived the direct and modal methods. Nelson[2] simplified the calculation of the direct method that yields an accurate solution. Rudisill[3] presented an iterative method. Ting[4] derived an accelerated subspace iteration to improve the efficiency. Wang,[5] by introducing the modal acceleration technique of dynamic response analysis, proposed two modified modal methods to improve the accuracy of the solution. Recently, Akgün[6] used multiple modal accelerations in non-self-adjoint systems with a singular coefficient matrix. Zeng[7] formulated a general expression for various modal methods, which is applied in viscous damping systems, and the convergence is speeded up with the use of shifted poles.

The modified modal method[5] uses the static solution to account for the contribution of truncated higher eigenvectors, which significantly improves the accuracy or effectiveness of the truncated modal superposition.[5,8] However, as a numerical approximate method, the accuracy is not completely controllable, and since the number of eigenvectors needed in obtaining the eigenvector derivatives considered cannot be known before performing the eigensolution, quite a large number of eigenvectors have to be calculated, which amounts to the most time-consuming portion of the computational effort. In this paper, we extended the modified modal method to an iterative procedure. Both of these concerns are addressed. The proposed method is efficient and numerically stable, and it converges to the accurate solution. A numerical example is presented to illustrate the performance of the method.

Modal Method

Consider a structural vibration system characterized by the real, symmetric stiffness matrix $K \in R^{n\times n}$, and mass matrix $M \in R^{n\times n}$, which are functions of the design variables. The eigenproblem is defined by

$$[K - \lambda_l M]\phi_l = 0 \tag{1}$$

in which λ_l and $\phi_l \in R^n$ are the lth eigenvalue and eigenvector, respectively. Given that the mass matrix is positive definite, ϕ_l can be normalized to satisfy

$$\phi_l^t M \phi_l = 1 \tag{2}$$

where t denotes the matrix transpose.

When λ_l is distinct, the differentiable eigenvector is uniquely defined by Eq. (1). Differentiating Eqs. (1) and (2) results in

$$[K - \lambda_l M]\phi_l' = -[K - \lambda_l M]'\phi_l = -[K' - \lambda_l' M - \lambda_l M']\phi_l \equiv F_l \tag{3}$$

$$\phi_l'^t M\phi_l + \phi_l^t M'\phi_l + \phi_l^t M\phi_l' = 0 \tag{4}$$

where the prime stands for the partial derivative with respect to the design variable. Premultiplying Eq. (3) by ϕ_l^t and taking advantage of $\phi_l^t[K - \lambda_l M] = 0$, one obtains[1]

$$\lambda_l' = \phi_l^t[K' - \lambda_l M']\phi_l \tag{5}$$

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The eigenvector derivative is then uniquely determined by Eqs. (3) and (4). The terms ϕ_l' can be represented as a linear combination of the complete eigenvectors,

$$\phi_l' = \sum_{i=1}^{n} c_{il}\phi_i \tag{6}$$

Substitution of Eq. (6) into Eqs. (3) and (4) yields[1]

$$c_{il} = \begin{cases} \dfrac{\phi_i^t F_l}{\lambda_i - \lambda_l} & i \neq l \\ -0.5\phi_l^t M'\phi_l & i = 1 \end{cases} \tag{7}$$

Usually, only m eigenvalues and eigenvectors are computed, $l < m \ll n$, and thus an approximate solution is obtained using the m available eigenvectors in the summation. Wang modified the truncated modal method by adding a residual static mode to approximate the contribution of the unavailable higher eigenvectors[5]:

$$\phi_l' \approx \sum_{i=1}^{m} c_{il}\phi_i + d_l w_l \tag{8}$$

where w_l is the residual static mode,

$$w_l = K^{-1}F_l - \sum_{i=1}^{m}\frac{\phi_i^t F_l \phi_i}{\lambda_i} = \sum_{i=m+1}^{n}\frac{\phi_i^t F_l \phi_i}{\lambda_i} \tag{9}$$

and d_l is the scalar coefficient,

$$d_l = \frac{w_l^t F_l}{w_l^t[K - \lambda_l M]w_l} \tag{10}$$

It was reported[5,8] (it also can be seen from the example presented in this paper) that Wang's solution significantly improves the accuracy or converges much faster than the superposition of the truncated modes.

Iterative Procedure

A simple, effective iterative procedure for calculating eigenvector derivatives is described herein. First, the modal matrix ψ is partitioned into

$$\psi = [\psi_1 \quad \psi_2] \tag{11}$$

where

$$\psi_1 = [\phi_1, \phi_2, \ldots, \phi_m] \tag{12}$$

$$\psi_2 = [\phi_{m+1}, \phi_{m+2}, \ldots, \phi_n] \tag{13}$$

Correspondingly, the eigenvalue matrix Λ is partitioned into

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \equiv \Lambda_1 + \Lambda_2 \tag{14}$$

where

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \tag{15}$$

$$\Lambda_2 = \mathrm{diag}(\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n) \tag{16}$$

Then the eigenvector derivative ϕ_l' is expressed as

$$\phi_l' = \phi_{l_1}' + \phi_{l_2}' = \psi_1 C_1 + \psi_2 C_2 \tag{17}$$

where

$$C_1 = \{c_{1l}, c_{2l}, \ldots, c_{ml}\}^t \tag{18}$$

$$C_2 = \{c_{m+1,l}, c_{m+2,l}, \ldots, c_{nl}\}^t \tag{19}$$

and

$$\phi_{l_1}' = \psi_1 C_1 = \sum_{i=1}^{m} c_{il}\phi_i \tag{20}$$

$$\phi_{l_2}' = \psi_2 C_2 = \sum_{i=m+1}^{n} c_{il}\phi_i \tag{21}$$

with the coefficients c_{il} given by Eq. (7). Note that ϕ'_{l_1} can be calculated by Eq. (18) directly, and only ϕ'_{l_2} remains to be considered herein because of the lack of the higher eigenvectors ψ_2.

Now we define a matrix

$$\Phi \equiv \psi_2 \Lambda_2^{-1} \psi_2^t \tag{22}$$

which is actually obtained from

$$\Phi = K^{-1} - \psi_1 \Lambda_1^{-1} \psi_1^t \tag{23}$$

The matrix Φ thus obtained satisfies $\Phi K \psi_1 = \Phi M \psi_1 = \Phi K \phi'_{l_1} = \Phi M \phi'_{l_1} = 0$, $\Phi K \phi'_{l_2} = \phi'_{l_2}$, and $\Phi M \phi'_{l_2} = \psi_2 \Lambda_2^{-1} C_2$. Premultiplication of Eq. (3) with Φ gives

$$\Phi[K - \lambda_l M]\phi'_l = \Phi[K - \lambda_l M]\phi'_{l_2} = \phi'_{l_2} - \lambda_l \Phi M \phi'_{l_2} = \Phi F_l \tag{24}$$

or

$$\phi'_{l_2} = \Phi F_l + \lambda_l \Phi M \phi'_{l_2} \tag{25}$$

The term ϕ'_{l_2} is uniquely defined by Eq. (25), and therefore it can be used as an iterative process to calculate ϕ'_{l_2}:

$$\left(\phi'_{l_2}\right)_q = \Phi F_l + \lambda_l \Phi M \left(\phi'_{l_2}\right)_{q-1} \qquad q = 1, 2, \ldots \tag{26}$$

Let the initial solution $(\phi'_{l_2})_0 = \{0\}$; one has

$$\left(\phi'_{l_2}\right)_1 = \Phi F_l = \phi'_{l_2} - \lambda_l \Phi M \phi'_{l_2} = \psi_2 C_2 - \lambda_l \psi_2 \Lambda_2^{-1} C_2$$
$$= \sum_{i=m+1}^{n} \left(1 - \frac{\lambda_l}{\lambda_i}\right) \frac{\phi_i^t F_l}{\lambda_i - \lambda_l} \phi_i \tag{27}$$

which, in fact, is the result of Wang's method with $d_l = 1$. From the comparison of Eq. (27) with the exact solution [Eqs. (21) and (7)], it is seen that λ_l/λ_i represents the error in the component of ϕ'_{l_2} because of the ith eigenvector. Substituting Eq. (27) back into Eq. (26) produces

$$\left(\phi'_{l_2}\right)_2 = \sum_{i=m+1}^{n} \left[1 - \left(\frac{\lambda_l}{\lambda_i}\right)^2\right] \frac{\phi_i^t F_l}{\lambda_i - \lambda_l} \phi_i \tag{28}$$

Since $i > l$, $(\lambda_l/\lambda_i) < 1$, and $(\lambda_l/\lambda_i)^2 < (\lambda_l/\lambda_i)$, the error is further reduced than that in the first iteration $(\phi'_{l_2})_1$. It can be shown that after q iterations

$$\left(\phi'_{l_2}\right)_q = \sum_{i=m+1}^{n} \left[1 - \left(\frac{\lambda_l}{\lambda_i}\right)^q\right] \frac{\phi_i^t F_l}{\lambda_i - \lambda_l} \phi_i \tag{29}$$

When $q \to \infty$, $(\lambda_l/\lambda_i)^q \to 0$, and $(\phi'_{l_2})_q$ will converge to the exact solution. Note that because of the transformation of Φ, all of the components due to the lower eigenvectors are eliminated from the iterative equation (26). Consequently, no round-off error will be accumulated, and the procedure is numerically stable.

Numerical Example

A small example serves to illustrate the performance of the proposed method. The nonzero elements of the 6 × 6 stiffness and mass matrices and their derivatives are listed in Table 1. The system eigenvalues are shown in Table 2, and the comparison of the first three eigenvector derivatives generated by simple superposition of the truncated eigenvectors, Wang's modified modal method, and the proposed procedure are given in Table 3.

In this example, four eigenvectors are used in the truncated modal superposition, whereas three are used for Wang's and the proposed methods. It is seen from Table 3 that Wang's solution significantly improves the truncated modal method. However, for the second and

Table 1 Elements of K, M, K', and M'

Location	K	M	K'	M'
(1, 1)	0.60e+6	0.20e+0	0.10e+5	0.10e+0
(1, 2)	−0.10e+6	0	−0.10e+5	0
(1, 3)	−0.20e+6	0	−0.10e+5	0
(2, 2)	0.40e+6	0.50e+0	0	0.20e+0
(2, 3)	−0.10e+6	0	−0.10e+5	0
(3, 3)	0.90e+6	0.30e+0	0.10e+5	0.20e+0
(3, 4)	−0.20e+6	0	0	0
(3, 5)	−0.10e+6	0	−0.10e+5	0
(4, 4)	0.70e+6	0.60e+0	0.10e+5	0.20e+0
(4, 5)	−0.20e+6	0	−0.10e+5	0
(4, 6)	−0.10e+6	0	−0.10e+5	0
(5, 5)	0.80e+6	0.50e+1	0.10e+5	0
(6, 6)	0.30e+6	0.20e+0	0.10e+5	0.20e+0

Table 2 System eigenvalues

0.1376e+6	0.6704e+6	0.9478e+6	0.1629e+7	0.2382e+7	0.3860e+7

Table 3 Comparison of eigenvector derivatives

	Exact	Simple modal superposition (four modes)	Error,[a] %	Wang's method[b] (three modes)	Error, %	Proposed method[c] (three modes)	Error, %
ϕ'_1	0.012385	0.009265	−25.19	0.012312	−0.59	0.012385	0.00
	0.017784	0.018951	6.56	0.017792	0.04	0.017784	0.00
	0.017728	0.011779	−33.56	0.017739	0.06	0.017728	0.00
	0.020676	0.022097	6.88	0.020674	−0.01	0.020676	0.00
	−0.004314	−0.004292	−0.52	−0.004314	0.00	−0.004314	0.00
	0.018962	0.018245	−3.78	0.018974	0.06	0.018962	0.00
ϕ'_2	−0.043908	−0.099177	125.88	−0.043326	−1.33	−0.043897	−0.03
	−0.294695	−0.280765	−4.73	−0.295473	0.26	−0.294701	0.00
	−0.033676	−0.087971	161.23	−0.030056	−10.75	−0.033659	−0.05
	−0.070563	−0.055782	−20.95	−0.068868	−2.40	−0.070538	−0.04
	−0.018790	−0.018588	−1.08	−0.018875	0.45	−0.018791	0.00
	0.045495	0.037455	−17.67	0.037307	−18.00	0.045397	−0.22
ϕ'_3	−0.057494	−0.093199	62.10	−0.047224	−17.86	−0.057223	−0.47
	0.071805	0.082889	15.44	0.066570	−7.29	0.071714	−0.13
	−0.146340	−0.197215	34.76	−0.118558	−18.99	−0.146060	−0.19
	−0.469460	−0.456694	−2.72	−0.472467	0.64	−0.469312	−0.03
	−0.009887	−0.009695	−1.94	−0.010090	2.05	−0.009895	0.08
	0.479234	0.472595	−1.39	0.470779	−1.76	0.478503	−0.15

[a]Defined by $100 \times [(\phi'_{il})_q - \phi'_{il}]/|\phi'_{il}|$, where ϕ'_{il} is the ith element of ϕ'_l.
[b]Equations (8), (9), and (10).
[c]Equation (26), $(\phi'_{l_2})_0 = \{0\}$. Iterations = 5.

third eigenvector derivatives, quite large errors are found at the first, third, and sixth degrees of freedom. In contrast, using the proposed method, all of the errors are less than 0.5% after five iterations.

Conclusion

The direct application of the truncated modal method needs a large number of eigenvectots to achieve good approximate eigenvector derivatives. Wang's method[5] requires substantially fewer eigenvectors and thus may significantly improve the effectiveness. However, since the number of eigenvectors needed in the method cannot be known a priori, a large number of eigenvectors will still be necessary to be calculated in practical application, which amounts to the main portion of the overall computational work. In addition, the solution accuracy, determined once the number of eigenvectors used in the method is chosen, cannot be controlled directly. These two deficiencies of Wang's method may hinder its application. In this paper, an iterative procedure, which is limited to one design variable and distinct eigenvalues, is developed by extension of Wang's method to solve these problems. The iterative process converges to the exact solution, and numerically, it is stable. The number of eigenvectors needed in the procedure can be just the same as that of the eigenvector derivatives to be calculated, although the calculation of several more eigenvectors is recommended.

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